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Note

Some families of chromatically unique bipartite graphs

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Abstract

A graph is said to be chromatically unique (or χ-unique) if it is uniquely determined by its chromatic polynomial. Let $K^{-r}(p, q)$ denote the family of graphs obtained from $K_{p,q}$ by deleting any r distinct edges. In this paper, we study the chromaticity of the graphs in $K^{-r}(p, q)$. A sufficient condition is given for a member of $K^{-r}(p, q)$ to be χ-unique and some families of χ-unique bipartite graphs are obtained. A conjecture is also proposed.

1. Introduction

We refer the reader to [3] for undefined terms in graph theory. A simple graph G is said to be chromatically unique (or simply χ-unique) if it is uniquely determined by its chromatic polynomial $P(G; t)$. In 1978, Chao and Whitehead [1] introduced and studied the notion of χ-unique graphs. Since then several families of graphs have been shown to be χ-unique (see [4, 5] for an account).

Let $K^{-r}(p, q)$ denote the family of graphs obtained from $K_{p,q}$ by deleting any r distinct edges. The following families of bipartite graphs are known to be χ-unique.

(a) $K_{p,q}$ where $q \geqslant p \geqslant 2$ [7],
(b) $K^{-1}(p, q)$ where $q \geqslant p \geqslant 3$ [7],
(c) $K^{-2}(p, p+h)$ where $h = 0, 1$, or 3, and $p \geqslant 4$ [2, 6],
(d) $K^{-3}(p, p+h)$ where $h = 0$ or 1, and $p + h \geqslant 5$ [6],
(e) $K^{-4}(p, p+1)$ where $p \geqslant 5$ [6],
(f) $K^{-r}(p, p+h)$ where $h = 0$ or 1, $p \geqslant 3$, $p \geqslant r$, and the r edges deleted are independent [2].

In [2, 6] it is also proved that for every $h \geqslant 2$, there exists an integer $p_0(h)$ such that for every $p \geqslant p_0(h)$, each member of $K^{-2}(p, p+h)$ is χ-unique.

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In this paper, we study the chromaticity of the graphs in $K^{-r}(p,q)$. A sufficient condition is given for a member of $K^{-r}(p,q)$ to be χ-unique. It is shown that each member of $K^{-3}(p,q)$, each member of $K^{-4}(p,q)$ and the six members of $K^{-r}(p,q)$ (namely, $K_{p,q}-H_i$ for $i=1,\dots,6$, as defined in Section 2) are χ-unique, where p, q and r satisfy certain conditions. A conjecture on χ-uniqueness of the graphs in $K^{-r}(p,q)$ is also proposed.

2. Some lemmas

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively, and let $e(G)=|E(G)|$. Also, let $Q(G)$ denote the number of cycles C_4 of the order 4 without chords in G.

The following three lemmas are known.

Lemma A (see, for example, Teo and Koh [7]). *Let G and H be graphs such that $P(G;t)=P(H;t)$. If G is bipartite, then H is bipartite, $|V(G)|=|V(H)|$, $e(G)=e(H)$ and $Q(G)=Q(H)$.*

Lemma B (Teo and Koh [7]). *If $q\geqslant p\geqslant 3$, then $K_{p,q}$ and $K^{-1}(p,q)$ are χ-unique.*

Lemma C (Giudici and Lima de sá [2]). *If $q\geqslant p>r\geqslant 2$, then $Q(G)>Q(K_{p,q}-rK_2)$ for each member G of $K^{-r}(p,q)$ with $G\neq K_{p,q}-rK_2$.*

To prove our main results we need additional notation and lemmas.

Let G be a subgraph of $K_{p,q}$. Denote $K_{p,q}-E(G)$ by $G^c(p,q)$ or simply G^c.

Given $q\geqslant p>r\geqslant 2$, we define six connected subgraphs S_r, T_r, N_r, Y, W and M of $K_{p,q}$ as follows:

S_r and T_r are isomorphic to $K_{1,r}$; W and M are isomorphic to the path P_5 of order 5; N_r has $r\geqslant 3$ edges and a vertex of degree $r-1$; Y has 4 edges and a vertex of degree 3. The vertex of degree r in S_r (resp. the vertices of degree 1 in T_r, a vertex of degree $r-1$ in N_r, the vertex of degree 2 in Y, the two vertices adjacent to end-vertices in W, the vertices of degree 1 in M) is in the partite set with q vertices of $K_{p,q}$.

Let $q\geqslant p>r\geqslant 2$, and let $H_1=H_1(r)=S_r$, $H_2=H_2(r)=T_r$, $H_3=H_3(r)=N_r$, $H_4=H_4(r)=S_2\cup(r-2)K_2$, $H_5=H_5(r)=T_2\cup(r-2)K_2$, $H_6=H_6(r)=rK_2$. Also, for $i=1,\dots,6$, let

$$h_i=Q(H_i^c)-\tfrac14 pq(p-1)(q-1)+rpq-rp-rq-\tfrac12 r(r-3).$$

It is not difficult to show the following.

Lemma 1. $h_1=r(r-1)(\frac12 q-1)$, $h_2=r(r-1)(\frac12 p-1)$, $h_3=(r-1)(r-2)(\frac12 q-1)+p-r$, $h_4=q-2$, $h_5=p-2$, $h_6=0$.

Corollary 1. *Let $q = p + h \geqslant p > r \geqslant 3$. Then*
(1) $Q(H_3^c) > Q(H_i^c)$ for $i = 4, 5, 6$.
(2) *If $h > 0$ and $p > \frac{1}{2}h(r-1)+1$, then $Q(H_i^c) > Q(H_{i+1}^c)$ for $i = 1, \ldots, 5$.*

Let $q \geqslant p > 4$, and let $G_1 = S_4$, $G_2 = T_4$, $G_3 = N_4$, $G_4 = C_4$, $G_5 = Y$, $G_6 = S_3 \cup K_2$, $G_7 = W$, $G_8 = M$, $G_9 = T_3 \cup K_2$, $G_{10} = 2S_2$, $G_{11} = S_2 \cup T_2$, $G_{12} = P_4 \cup K_2$, $G_{13} = 2T_2$, $G_{14} = S_2 \cup 2K_2$, $G_{15} = T_2 \cup 2K_2$, $G_{16} = 4K_2$. Also, for $i = 1, \ldots, 16$, let

$$g_i = Q(G_i^c) - \tfrac{1}{4}pq(p-1)(q-1) + 4pq - 4p - 4q - 2.$$

It is not difficult to show the following.

Lemma 2. $g_1 = 6q - 12$, $g_2 = 6p - 12$, $g_3 = p + 3q - 10$, $g_4 = 2p + 2q - 11$, $g_5 = 3p + q - 10$, $g_6 = 3q - 6$, $g_7 = p + 2q - 8$, $g_8 = 2p + q - 8$, $g_9 = 3p - 6$, $g_{10} = 2q - 4$, $g_{11} = p + q - 4$, $g_{12} = p + q - 5$, $g_{13} = 2p - 4$, $g_{14} = q - 2$, $g_{15} = p - 2$, $g_{16} = 0$.

Corollary 2. *Let $q = p + h \geqslant p > 4$.*
(1) *If $q > p$, then $g_3 > g_i$ and $Q(G_3^c) > Q(G_i^c)$ for $i = 4, \ldots, 16$. If $q = p$, then $g_3 = g_5 > g_i$ for $i = 4, 6, \ldots, 16$.*
(2) *If $p > h + 2$, then $g_i > g_{14}$ for $i = 1, \ldots, 13$.*
(3) *If $p \geqslant 2h + 5 \geqslant 11$, then $g_i > g_{i+1}$ for $i = 1, \ldots, 15$.*

We denote by $d_G(v)$ the degree of a vertex v in G, by $N(v)$ the set of vertices adjacent to v in G.

Lemma 3. *Let G be a subgraph of $K_{p,q}$ and $f = (u, v)$ be an edge of G, where vertex v is in the partite set with q vertices of $K_{p,q}$. Let $a = d_G(v)$, $b = d_G(u)$, $s = e(G - N(u) - N(v))$ where $0 \leqslant s \leqslant e(G) - (a + b - 1)$. Then*

$$Q(G-f)^c = Q(G^c) + (p-a)(q-b) - s.$$

Proof. The set of C_4's of the graph $(G-f)^c$ is the union of two subsets A and B, where each C_4 in A does not contain f and each C_4 in B contains f. It is easy to see that $|A| = Q(G^c)$ and $|B| = (p-a)(q-b) - s$, and the result follows. □

Lemma 4. *Let G be a subgraph of $K_{p,q}$, where $q = p + h > p > e(G) = r \geqslant 3$ and $p > h + 2$. If G is not isomorphic to H_4, H_5 or H_6, then $Q(G^c) > Q(H_4^c)$.*

Proof. We proceed by induction on r.

It is easy to show that

$$K^{-3}(p,q) = \{H_i^c(3) \mid i = 1, \ldots, 6\},$$

$$K^{-4}(p,q) = \{G_i^c \mid i = 1, \ldots, 16\}.$$

By Corollaries 1 and 2, the conclusion for $r = 3$ or 4 is true.

Suppose that the conclusion for $r = k \geqslant 4$ is true. Let $e(G) = k + 1 \geqslant 5$. Since G is not isomorphic to $H_4(k+1)$, $H_5(k+1)$ or $H_6(k+1)$, it is easy to see that there exists an edge $f = (u, v)$ in G such that $G - f$ is also not isomorphic to $H_4(k)$, $H_5(k)$ or $H_6(k)$. By the induction hypothesis,

$$Q(G-f)^c > Q(S_2 \cup (k-2)K_2)^c.$$

By Lemma 3, we have

$$Q(G^c) + (p-a)(q-b) - s > Q(S_2 \cup (k-1)K_2)^c + (p-1)(q-1) - k. \tag{1}$$

Let $d = (a-1)(q-b) + (b-1)(p-1)$. If $a = b = 1$, it is clear that $s = k$, then $d = 0 = k - s$. If $ab \geqslant 2$, then since $p > k + 1$, it follows that $d \geqslant p - 1 > k \geqslant k - s$. The inequality $d \geqslant k - s$ implies that

$$(p-1)(q-1) - k \geqslant (p-a)(q-b) - s.$$

By the inequality (1), the conclusion for $r = k + 1$ is true.

This completes the proof by induction. $\square$

Lemma 5. *Let G be a subgraph of $K_{p,q}$, where $q \geqslant p > e(G) = r \geqslant 3$, such that G^c is not isomorphic to S_r^c, T_r^c or N_r^c. Then $Q(N_r^c) > Q(G^c)$.*

Proof. Suppose that H is a subgraph of $K_{p,q}$ with $e(H) = r$, we say that H has Property P if H^c is not isomorphic to S_r^c, T_r^c or N_r^c.

We proceed by induction on r.

By Corollaries 1 and 2, the lemma for $r = 3$ or 4 is true.

Suppose that the lemma for $r = k \geqslant 4$ is true. Let $e(G) = k + 1 \geqslant 5$. Since G has Property P, it is easy to see that there exists an edge $f = (u, v)$ in G such that the graph $G - f$ has also Property P. By the induction hypothesis, $Q(N_k^c) > Q(G-f)^c$. By Lemma 3, we have

$$Q(N_{k+1}^c) + (p-k)(q-1) > Q(G^c) + (p-a)(q-b) - s.$$

To show that $Q(N_{k+1}^c) > Q(G^c)$, it is sufficient to show that

$$(p-a)(q-b) - s \geqslant (p-k)(q-1),$$

i.e.,

$$(k-a)(q-1) \geqslant (b-1)(p-a) + s. \tag{2}$$

Suppose that $e(G) = k + 1 = a + b - 1 \geqslant 5$. G has Property P, so $b \geqslant 3$ and $a \geqslant 2$. If $a = b = 3$, then $Q(N_5^c) > Q(G^c)$. If $b = 3$ and $a \geqslant 4$ or $b \geqslant 4$ and $a \geqslant 2$, then there exists an edge $f' = (u', v') \neq f$ in G such that $G - f'$ has Property P and $k + 1 \geqslant a' + b'$, where $a' = d_G(v')$ and $b' = d_G(u')$. Therefore, without loss of generality, we can suppose that $e(G) = k + 1 \geqslant a + b$. It is easy to check that

$$Q(N_{k+1}^c) > Q(S_k \cup K_2)^c. \tag{3}$$

We consider three cases:

Case 1: $s \leqslant k - a$ *and* $a \geqslant 2 - (q - p)$. Then $q - 2 \geqslant p - a > k - a \geqslant 0$, and so

$$(k - a)(q - 1) \geqslant (k - a)(q - 2) + s \geqslant (b - 1)(p - a) + s.$$

Case 2: $s \leqslant k - a$ *and* $1 \leqslant a < 2 - (q - p)$. Then $p = q$ and $a = 1$. Since G has Property P and the inequality (3) holds, we can suppose that $k > b$. Thus,

$$\begin{aligned}(k - a)(q - 1) &= (k - b)(p - 1) + (b - 1)(p - 1)\\ &> k - 1 + (b - 1)(p - a) \geqslant s + (b - 1)(p - a).\end{aligned}$$

Case 3: $s \geqslant k + 1 - a$. Since

$$s \leqslant e(G) - (a + b - 1) = k + 1 - (a + b - 1) \leqslant k + 1 - a,$$

we have $s = k + 1 - a$ and $b = 1$. Since G has Property P and inequality (3) holds, we can suppose that $s > 1$, i.e., $k - a > 0$. Thus,

$$(k - a)(q - 1) \geqslant k - a + 1 = s = s + (b - 1)(p - a).$$

So inequality (2) holds. This completes the proof by induction. □

Corollary 3. *If* $q \geqslant p > r \geqslant 2$, *then* $Q(K_{p,q} - S_r) > Q(G)$ *for each member* G *of* $K^{-r}(p, q)$ *with* $G \neq K_{p,q} - S_r$.

Proof. By Lemma 1, $h_1 \geqslant h_2 > h_6$. Thus, $Q(S_r^c) \geqslant Q(T_r^c) > Q(rK_2)^c$, and the equality holds if and only if $q = p$. If $r \geqslant 3$, then $h_1 > h_3$ and $Q(S_r^c) > Q(N_r^c)$. The result now follows by Lemma 5. □

Let $G \in K^{-r}(p, q)$, where

$$q = p + h > p > h + 2 \geqslant r \geqslant 1.$$

If $1 \leqslant c < p$, then $e(K^{-1}(p - c, q + c)) \leqslant e(G)$, and the equality holds if and only if $c = 1$ and $r = h + 2$. If $H \in K^{-e}(p - c, q + c)$ where $e \geqslant 0$, then by Lemmas A and B, we have $p(H; t) \neq p(G; t)$.

Let $1 \leqslant a \leqslant \frac{1}{2}h$, $1 \leqslant r \leqslant h + 2$ and $p > \frac{1}{4}h^2 + h + 1$. Then

$$r + a(h - a) \leqslant h + 2 + \tfrac{1}{4}h^2 < p + a \leqslant p + h - a = q - a$$

and

$$e(K_{p+a,q-a}) - (r + a(h-a)) = pq - r = e(G).$$

Lemma 6. *Let* $q = p + h$, $1 \leqslant a \leqslant \frac{1}{2}h$, $1 \leqslant r \leqslant h + 2$, *and* $p \geqslant \frac{8}{27}h^2 + \frac{1}{3}h + 5r + 6$. *Also, let*

$$F(a) = Q(K_{p+a,q-a} - S_{r+a(h-a)}) - Q(K_{p,q} - rK_2).$$

Then $F(a) < 0$.

Proof. Computing $F(a)$, we have

$$4F(a) = (2a - 2h)ap^2 + f_1(a)p + f_2(a),$$

where

$$\begin{aligned} f_1(a) &= 2a^2(h-a)^2 + 2a^2(h - r - 2) + ah(4r - 2h + 4) + 2r(r - a^2 - 1) \\ &< a2a(h-a)(h-a) + ah(h - r + 2) + ah(4r - 2h + 4) + 2arh \\ &\leqslant ah(\tfrac{8}{27}h^2 - h + 5r + 6), \\ f_2(a) &= a^2(h-a)^2(2h - 2a - 5) + ah^2(4r + 1) + a^2((4r - 2)a \\ &\quad - 8hr + h) - a(h-a)(8r + 1) - 2ar(r-1) + (2h - 4)r(r-1) \\ &< \tfrac{2}{3}a3a(h-a)(h-a)(h-a-1) - 3a(h-a)^2 \\ &\quad + ah^2(4r+1) + 2ar(r-1)(h-2) \\ &< \tfrac{2}{3}a(\tfrac{3}{4}h - \tfrac{1}{4})^4 - \tfrac{3}{4}ah^2 + ah^2(4r+1) + 2ah^2(h+1) \\ &< ah^2(\tfrac{27}{128}h^2 - \tfrac{9}{32}h + \tfrac{9}{64}) + ah^2(2h + 4r + \tfrac{9}{4}) \\ &< ah\tfrac{4}{3}h(\tfrac{81}{512}h^2 + \tfrac{165}{128}h + 3r + 2). \end{aligned}$$

Since

$$p \geqslant \tfrac{8}{27}h^2 + \tfrac{1}{3}h + 5r + 6 > \tfrac{81}{512}h^2 + \tfrac{165}{128}h + 3r + 2,$$

it follows that

$$-4F(a)/(ah) > p^2 - (\tfrac{8}{27}h^2 - h + 5r + 6)p - \tfrac{4}{3}h(\tfrac{81}{512}h^2 + \tfrac{165}{128}h + 3r + 2) > 0,$$

and so $F(a) < 0$. □

3. Main results

In this section, we present our main results. Let h and r be nonnegative integers, and let

$$p_0(h, r) = \tfrac{8}{27}h^2 + \tfrac{1}{3}h + 5r + 6.$$

Theorem 1. *Let G be a given member of $K^{-r}(p, q)$, where*

$$q = p + h > p \geqslant p_0(h, r) > h + 2 \geqslant r \geqslant 2.$$

If $P(H; t) \neq P(G; t)$ (in particular, $Q(H) \neq Q(G)$) for each member H of $K^{-r}(p, q)$ with $H \neq G$, then G is χ-unique.

Proof. As proved above, if $H_0 \in K^{-e}(p - c, q + c)$, where $1 \leqslant c < p$ and $e \geqslant 0$, then $P(H_0; t) \neq P(G; t)$. If $H_0 \in K^{-e}(p + a, q - a)$ where $1 \leqslant a \leqslant \frac{1}{2}h$ and $e = r + a(h - a)$, then by Corollary 3, Lemmas 6 and C, we have

$$Q(H_0) \leqslant Q(K_{p+a,q-a} - S_{r+a(h-a)}) < Q(K_{p,q} - rK_2) \leqslant Q(G).$$

Thus, $P(H_0; t) \neq P(G; t)$ by Lemma A. Since $P(H; t) \neq P(G; t)$ for each member H of $K^{-r}(p, q)$ with $H \neq G$, it follows that G is χ-unique. $\square$

Theorem 2. *Let $H_i, 1 \leqslant i \leqslant 6$, denote the graph defined as above, and let $3 \leqslant r \leqslant h + 2$. If*

$$p \geqslant \max\{p_0(h, r), \tfrac{1}{2}h(r - 1) + \tfrac{3}{2}\},$$

then the graph $K_{p,p+h} - H_i$ is χ-unique.

Proof. By Corollary 1, $Q(H_i^c) > Q(H_{i+1}^c)$ for $i = 1, \ldots, 5$. Suppose that G is a subgraph of $K_{p,p+h}$ with $e(G) = r \geqslant 4$, and G^c is not isomorphic to H_i^c for $i = 1, \ldots, 6$. By Lemmas 4 and 5, $Q(H_3^c) > Q(G^c) > Q(H_4^c)$. By Theorem 1, the conclusion follows. $\square$

Theorem 3. *If $p \geqslant p_0(h, 3)$, then each member of $K^{-3}(p, p + h)$ is χ-unique.*

Proof. It is known [6] that each member of $K^{-3}(p, p)$ with $p \geqslant 5$ is χ-unique. If $h \geqslant 1$, then the theorem is a corollary of Theorem 2. $\square$

Theorem 4. *If $h \geqslant 3$ and $p \geqslant p_0(h, 4)$, then each member of $K^{-4}(p, p + h)$ is χ-unique.*

Proof. By Corollary 2 and Theorem 1, the conclusion follows. $\square$

Remark. If r is not large (e.g., $r = 5$), then it is not difficult to compute $Q(G)$ for each member G of $K^{-r}(p, q)$. By Theorem 1, one can obtain some additional χ-unique bipartite graphs.

To conclude this paper, we would like to propose the following.

Conjecture. For every $r \geqslant 0$ and every $h \geqslant h_0(r)$, where $h_0(r)$ depends on r, there is an integer $p_1(h, r)$ depending on h and r, such that for every $p \geqslant p_1(h, r)$, each member of $K^{-r}(p, p + h)$ is χ-unique.

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